Relations and functions.

- Given two non-empty sets P and Q. The Cartesian product $P \times Q$ is the set of all ordered pairs of elements from P and Q, i.e., $P \times Q = \{ (p,q) : p \in P, q \in Q \}$ If either P or Q is the null set, then $P \times Q$ will also be empty set, i.e., $P \times Q = \varphi$.

  - ➢ Two ordered pairs are equal, if and only if the corresponding first elements are equal and the second elements are also equal.
  - ➢ If there are p elements in A and q elements in B, then there will be pq elements in $A \times B$, i.e., if $n(A) = p$ and $n(B) = q$, then $n(A\times B) = pq$.
  - ➢ If A and B are non-empty sets and either A or B is an infinite set, then so is $A \times B$.
  - ➢ $A \times A \times A = \{(a, b, c) : a, b, c \in A\}$. Here (a, b, c) is called an ordered triplet.

- A relation R from a non-empty set A to a non-empty set B is a subset of the cartesian product $A \times B$. The subset is derived by describing a relationship between the first element and the second element of the ordered pairs in $A \times B$. The second element is called the image of the first element.
- The set of all first elements of the ordered pairs in a relation R from a set A to a set B is called the domain of the relation R.
- The set of all second elements in a relation R from a set A to a set B is called the range of the relation R. The whole set B is called the codomain of the relation R.
- A relation f from a set A to a set B is said to be a function if every element of set A has one and only one image in set B. If f is a function from A to B and $(a, b) \in f$, then $f(a) = b$, where b is called the image of a under f and a is called the pre-image of b under f.
- Addition of two real functions Let $f: X \rightarrow R$ and $g: X \rightarrow R$ be any two real functions, where $X \subset R$. Then, we define $(f + g): X \rightarrow R$ by $(f + g)(x) = f(x) + g(x)$, for all $x \in X$.

- Subtraction of a real function from another Let $f : X \to R$ and $g: X \to R$ be any two real functions, where $X \subset R$. Then, we define $(f - g) : X \to R$ by $(f - g)(x) = f(x) - g(x)$, for all $x \in X$.
- Multiplication by a scalar Let $f : X \to R$ be a real valued function and α be a scalar. Here by scalar, we mean a real number. Then the product αf is a function from X to R defined by $(\alpha f)(x) = \alpha f(x)$, $x \in X$.
- Multiplication of two real functions. The product (or multiplication) of two real functions $f: X \to R$ and $g: X \to R$ is a function $fg: X \to R$ defined by $(fg)(x) = f(x) g(x)$, for all $x \in X$. This is also called point wise multiplication.

Examples

- Let $A = \{1,2,3\}$ $B = \{3,4\}$ and $C = \{4,5,6\}$. Find (i) $A \times (B \cap C)$ (ii) $(A \times B) \cap (A \times C)$

Solution:-

(i) By the definition of the intersection of two sets, $(B \cap C) = \{4\}$.

Therefore, $A \times (B \cap C) = \{(1,4), (2,4), (3,4)\}$.

(ii) Now $(A \times B) = \{(1,3), (1,4), (2,3), (2,4), (3,3), (3,4)\}$

and $(A \times C) = \{(1,4), (1,5), (1,6), (2,4), (2,5), (2,6), (3,4), (3,5), (3,6)\}$

- Let A = {1, 2} and B = {3, 4}. Find the number of relations from A to B.

  Solution:-

  A × B = {(1, 3), (1, 4), (2, 3), (2, 4)}.

  Since n (A×B) = 4, the number of subsets of A×B is 2^4. Therefore, the number of relations from A into B will be 2^4.

- Let N be the set of natural numbers and the relation R be defined on N such that R = {(x, y) : y = 2x, x, y ∈ N}. What is the domain, codomain and range of R? Is this relation a function?

  Solution:-

  The domain of R is the set of natural numbers N. The codomain is also N.

  The range is the set of even natural numbers.

  Since every natural number n has one and only one image, this relation is a function.

- Let $f = \{(1,1), (2,3), (0, -1), (-1, -3)\}$ be a linear function from Z into Z. Find $f(x)$.

  Solution:-

  Since f is a linear function, $f(x) = mx + c$. Also, since $(1, 1), (0, -1) \in R$,

  $f(1) = m + c = 1$ and $f(0) = c = -1$. This gives $m = 2$ and $f(x) = 2x - 1$.